COMPLEX INTERPOLATION OF Z-SPACES

ALEX AMENTA

Abstract. We prove that the Z-spaces $Z_s^{p,q}$ form a complex interpolation scale for all $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, filling a gap in recent work with Pascal Auscher.

Introduction

This note is an addendum to the recent monograph [2], in which a full complex interpolation result for the so-called 'Z-spaces' was conspicuously absent. Here we prove the missing result, and indicate how the results of [2] are affected.

1. Multiplication, factorisation, and interpolation

First we set up some notation. Fix a natural number $n \geq 1$ and define the upper half-space $\mathbb{R}^{1+n}_+ := (0,\infty) \times \mathbb{R}^n$. The set of Lebesgue measurable functions on $\mathbb{R}^{1+n}_+$, defined modulo sets of measure zero, is denoted $L^0(\mathbb{R}^{1+n}_+)$.

Let $\mathcal{Q} = \{2^k((0,1)^n + j) : k \in \mathbb{Z}, j \in \mathbb{Z}^n\}$ denote the set of standard open dyadic cubes in $\mathbb{R}^n$. For each $Q \in \mathcal{Q}$ let $\ell(Q)$ denote the side length of Q, and define $\overline{Q} := Q \times (\ell(Q), 2\ell(Q)) \subset \mathbb{R}^{1+n}_+$. Then the *Whitney grid* $\mathcal{G} := \{\overline{Q} : Q \in \mathcal{Q}\}$ partitions $\mathbb{R}^{1+n}_+$ up to a set of measure zero. For a measurable function $f \in L^0(\mathbb{R}^{1+n}_+)$ write

$$[f]_{\overline{Q}} = \fint\!\!\!\fint_{\overline{Q}} f(t,y)\,dt\,dy$$

for the average of f on $\overline{Q}$, with respect to the measure $dt\,dy$.

Definition 1.1. For $p, q \in (0,\infty]$ and $s \in \mathbb{R}$, we define the *Z-space* $Z_s^{p,q}$ to be the set of all $f \in L^0(\mathbb{R}^{1+n}_+)$ such that the quasinorm

$$\begin{aligned}\|f\|_{Z_s^{p,q}} &:= \|\ell(Q)^{-s}[|f|^q]_{\overline{Q}}^{1/q}\|_{\ell^p(\mathcal{G},\ell(Q)^n)} \\ &= \left(\sum_{\overline{Q}\in\mathcal{G}} \ell(Q)^n \left| \ell(Q)^{-s}[|f|^q]_{\overline{Q}}^{1/q} \right|^p \right)^{1/p}\end{aligned}$$

is finite, with the obvious modifications when p or q is infinite.

This is equivalent to the definitions given in [2, §2.2] and [1]. Our goal is to prove the following complex interpolation theorem.

Theorem 1.2. Suppose $s_0, s_1 \in \mathbb{R}$, $p_0, p_1, q_0, q_1 \in (0,\infty)$, and $\theta \in (0,1)$. Then

$$[Z_{s_0}^{p_0,q_0}, Z_{s_1}^{p_1,q_1}]_\theta = Z_{s_\theta}^{p_\theta,q_\theta},$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$, and $s_\theta = (1-\theta)s_0 + \theta s_1$.

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This was conjectured by Barton and Mayboroda, who first introduced the spaces $Z_s^{p,q}$ (which they denoted by $L(p, s+1, q)$) in the context of boundary value problems on $\mathbb{R}^{1+n}_+$ with boundary data in Besov spaces, and who proved Theorem 1.2 when $p_0, p_1 \geq 1$ and $q_0 = q_1 \geq 1$. [3, Theorem 4.13]. Here we use the complex interpolation method discussed by Kalton and Mitrea [6, §3], which is well-defined for all quasi-Banach interpolation couples, and which agrees with the usual (Calderón) complex interpolation method on Banach interpolation couples. We will use the following 'Calderón product formula', which is a special case of [6, Theorem 3.4].

Theorem 1.3 (Kalton–Mitrea)**.** Let X_0, X_1 be a pair of quasi-Banach function spaces on $\mathbb{R}^{1+n}_+$. Suppose that both X_0 and X_1 are A-convex and separable. Then

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta,$$

where $X_0^{1-\theta} X_1^\theta$ is the quasi-Banach function space consisting of those $h \in L^0(\mathbb{R}^{1+n}_+)$ such that the quasinorm

$$\|h\|_{X_0^{1-\theta} X_1^\theta} := \inf\{\|f\|_{X_0}^{1-\theta} \|g\|_{X_1}^\theta : |h| \leq |f|^{1-\theta}|g|^\theta,\ f \in X_0,\ g \in X_1\}$$

is finite, with the usual convention that the infimum of an empty set is $+\infty$.

A quasi-Banach function space X is A-convex if and only if it is lattice r-convex for some $r > 0$ (see [7, Theorem 2.2] and [8, Theorem 4.4]), which means that for all finite collections $f_1, \ldots, f_k \in X$

$$\Big\| \Big(\sum_{i=1}^k |f_i|^r \Big)^{1/r} \Big\|_X \lesssim \Big(\sum_{i=1}^k \|f_i\|_X^r \Big)^{1/r}.$$

It is easy to show that $Z_s^{p,q}$ is $\min(p,q)$-convex, hence A-convex. See [6] for further discussion of A-convexity.

In order to apply Theorem 1.3 to the Z-space scale, we need to establish multiplication and factorisation results, and before that we must introduce some terminology.

Definition 1.4. Let $s_0, s_1 \in \mathbb{R}$ and $p_0, p_1, q_0, q_1 \in (0, \infty]$. We write

$$Z_s^{p,q} \leftrightarrow Z_{s_0}^{p_0,q_0} \cdot Z_{s_1}^{p_1,q_1}$$

to mean that

- (multiplication property) if $f \in Z_{s_0}^{p_0,q_0}$ and $g \in Z_{s_1}^{p_1,q_1}$, then $fg \in Z_s^{p,q}$ with

$$\|fg\|_{Z_s^{p,q}} \lesssim \|f\|_{Z_{s_0}^{p_0,q_0}} \|g\|_{Z_{s_1}^{p_1,q_1}}, \tag{1}$$

  and
- (factorisation property) if $h \in Z_s^{p,q}$ then there exist $F \in Z_{s_0}^{p_0,q_0}$ and $G \in Z_{s_1}^{p_1,q_1}$ such that $h = FG$ and

$$\|F\|_{Z_{s_0}^{p_0,q_0}} \|G\|_{Z_{s_1}^{p_1,q_1}} \lesssim \|h\|_{Z_s^{p,q}}. \tag{2}$$

We abbreviate the multiplication and factorisation properties as

$$Z_s^{p,q} \leftarrow Z_{s_0}^{p_0,q_0} \cdot Z_{s_1}^{p_1,q_1} \quad \text{and} \quad Z_s^{p,q} \rightarrow Z_{s_0}^{p_0,q_0} \cdot Z_{s_1}^{p_1,q_1}$$

respectively.

This notation is easily extended to more than two factors, and formal computations involving commutativity and associativity (as in the proof of Proposition 1.8) are valid.

Remark 1.5. In all the cases that we cover here, the implicit constants in (1) and (2) can be taken to be 1. This is due to our 'dyadic' definition of the $Z^{p,q}_s$-quasinorm, and of course need not be true for other equivalent quasinorms.

Proposition 1.6 (Multiplication). *Suppose $s_0, s_1 \in \mathbb{R}$ and $p_0, p_1, q_0, q_1 \in (0, \infty]$, and let*

$$\frac{1}{p} := \frac{1}{p_0} + \frac{1}{p_1}, \quad \frac{1}{q} := \frac{1}{q_0} + \frac{1}{q_1}, \quad s := s_0 + s_1.$$

Then $Z^{p,q}_s \leftarrow Z^{p_0,q_0}_{s_0} \cdot Z^{p_1,q_1}_{s_1}$.

Proof. Let $f \in Z^{p_0,q_0}_{s_0}$ and $g \in Z^{p_1,q_1}_{s_1}$. Using Hölder's inequality twice and the assumptions on the exponents we have

$$\begin{aligned}
&\|\ell(Q)^{-s}[|fg|^q]^{1/q}_{\overline{Q}}\|_{\ell^p(\mathcal{G},\ell(Q)^n)} \\
&\leq \|\ell(Q)^{-s_0}[|f|^{q_0}]^{1/q_0}_{\overline{Q}}\ell(Q)^{-s_1}[|g|^{q_1}]^{1/q_1}_{\overline{Q}}\|_{\ell^p(\mathcal{G},\ell(Q)^n)} \\
&\leq \|\ell(Q)^{-s_0}[|f|^{q_0}|^{1/q_0}_{\overline{Q}}\|_{\ell^{p_0}(\mathcal{G},\ell(Q)^n)}\|\ell(Q)^{-s_1}[|f|^{q_1}|^{1/q_1}_{\overline{Q}}\|_{\ell^{p_1}(\mathcal{G},\ell(Q)^n)} \\
&= \|f\|_{Z^{p_0,q_0}_{s_0}}\|g\|_{Z^{p_1,q_1}_{s_1}},
\end{aligned}$$

proving the multiplication property. □

Lemma 1.7 (Single-exponent factorisation). *Suppose $p_0, p_1 \in (0, \infty]$ and $s_0, s_1 \in \mathbb{R}$, and let*

$$\frac{1}{p} := \frac{1}{p_0} + \frac{1}{p_1} \quad \textit{and} \quad s := s_0 + s_1.$$

Then

$$Z^{p,\infty}_s \to Z^{p_0,\infty}_{s_0} \cdot Z^{p_1,\infty}_{s_1} \quad \textit{and} \quad Z^{\infty,p}_s \to Z^{\infty,p_0}_{s_0} \cdot Z^{\infty,p_1}_{s_1}.$$

Proof. First suppose $f \in Z^{p,\infty}_s$; we may assume without loss of generality that f is nonnegative. Let

$$F := \sum_{\overline{Q} \in \mathcal{G}} \mathbf{1}_{\overline{Q}}\ell(Q)^{-s\frac{p}{p_0}+s_0} f^{p/p_0} \quad \text{and} \quad G := \sum_{\overline{Q} \in \mathcal{G}} \mathbf{1}_{\overline{Q}}\ell(Q)^{s\frac{p}{p_0}-s_0} f^{1-p/p_0}, \tag{3}$$

so that $FG = f$. A straightforward estimate gives

$$\|F\|_{Z^{p_0,\infty}_{s_0}} \leq \|f\|^{p/p_0}_{Z^{p,\infty}_s} \quad \text{and} \quad \|G\|_{Z^{p_1,\infty}_{s_1}} \leq \|f\|^{p/p_1}_{Z^{p,\infty}_s},$$

and so

$$\|F\|_{Z^{p_0,\infty}_{s_0}}\|G\|_{Z^{p_1,\infty}_{s_1}} \leq \|f\|_{Z^{p,\infty}_s}.$$

Now assume $g \in Z^{\infty,p}_s$ is nonnegative and define F' and G' as in (3), but with f replaced by g. The same argument as before yields

$$\|F'\|_{Z^{\infty,p_0}_{s_0}}\|G'\|_{Z^{\infty,p_1}_{s_1}} \leq \|g\|_{Z^{\infty,p}_s},$$

completing the proof. □

Proposition 1.8 (Factorisation). *Suppose $s_0, s_1 \in \mathbb{R}$ and $p_0, p_1, q_0, q_1 \in (0, \infty]$, and let*

$$\frac{1}{p} := \frac{1}{p_0} + \frac{1}{p_1}, \quad \frac{1}{q} := \frac{1}{q_0} + \frac{1}{q_1}, \quad s := s_0 + s_1.$$

Then

$$Z^{p,q}_s \to Z^{p_0,q_0}_{s_0} \cdot Z^{p_1,q_1}_{s_1}.$$

Proof. It suffices to show that

$$Z_s^{p,q} \to Z_{s_0}^{\infty,q} \cdot Z_{s_1}^{p,\infty}, \tag{4}$$

because by Proposition 1.6 and Lemma 1.7 along with (4) we have

$$\begin{aligned} Z_s^{p,q} &\to Z_{s/2}^{p,\infty} \cdot Z_{s/2}^{\infty,q} \\ &\to Z_{s_0/2}^{p_0,\infty} \cdot Z_{s_1/2}^{p_1,\infty} \cdot Z_{s_0/2}^{\infty,q_0} \cdot Z_{s_0/2}^{\infty,q_1} \\ &= Z_{s_0/2}^{p_0,\infty} \cdot Z_{s_0/2}^{\infty,q_0} \cdot Z_{s_1/2}^{p_1,\infty} \cdot Z_{s_1/2}^{\infty,q_1} \\ &\to Z_{s_0}^{p_0,q_0} \cdot Z_{s_1}^{p_1,q_1}. \end{aligned}$$

We now prove (4). Given $h \in Z_s^{p,q}$, let

$$F := \sum_{\overline{Q} \in \mathcal{G}} \mathbf{1}_{\overline{Q}} \ell(Q)^{s_0} h / [|h|^q]_{\overline{Q}}^{1/q} \quad \text{and} \quad G := \sum_{\overline{Q} \in \mathcal{G}} \mathbf{1}_{\overline{Q}} \ell(Q)^{-s_0} [|h|^q]_{\overline{Q}}^{1/q},$$

so that $FG = h$. We immediately have

$$\|F\|_{Z_{s_0}^{\infty,q}} = \sup_{\overline{Q} \in \mathcal{G}} \ell(Q)^{-s_0} \ell(Q)^{s_0} = 1$$

and

$$\begin{aligned} \|G\|_{Z_{s_1}^{p,\infty}} &= \|\ell(Q)^{-s_1} \ell(Q)^{-s_0} [|h|^q]_{\overline{Q}}^{1/q}\|_{\ell^p(\mathcal{G}, \ell(Q)^n)} \\ &= \|\ell(Q)^{-s} [|h|^q]_{\overline{Q}}^{1/q}\|_{\ell^p(\mathcal{G}, \ell(Q)^n)} \\ &= \|h\|_{Z_s^{p,q}}, \end{aligned}$$

proving (4). □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Since $Z_{s_0}^{p_0,q_0}$ and $Z_{s_1}^{p_1,q_1}$ are A-convex and separable, by Theorem 1.3 we have

$$[Z_{s_0}^{p_0,q_0}, Z_{s_1}^{p_1,q_1}]_\theta = (Z_{s_0}^{p_0,q_0})^{1-\theta} (Z_{s_1}^{p_1,q_1})^\theta.$$

We will show that

$$(Z_{s_0}^{p_0,q_0})^{1-\theta} (Z_{s_1}^{p_1,q_1})^\theta = Z_{s_\theta}^{p_\theta,q_\theta}. \tag{5}$$

This follows from Propositions 1.6 and 1.8 in a standard way, but we include the argument for completeness. Suppose that $h \in L^0(\mathbb{R}_+^{1+n})$ with $|h| \le |f|^{1-\theta} |g|^\theta$ for some $f \in Z_{s_0}^{p_0,q_0}$, $g \in Z_{s_1}^{p_1,q_1}$. Then by Proposition 1.6 we have

$$\begin{aligned} \|h\|_{Z_{s_\theta}^{p_\theta,q_\theta}} &\le \||f|^{1-\theta} |g|^\theta\|_{Z_{s_\theta}^{p_\theta,q_\theta}} \\ &\le \||f|^{1-\theta}\|_{Z_{s_0(1-\theta)}^{p_0/(1-\theta),q_0/(1-\theta)}} \||g|^\theta\|_{Z_{s_1\theta}^{p_1/\theta,q_1/\theta}} \\ &= \|f\|_{Z_{s_0}^{p_0,q_0}}^{1-\theta} \|g\|_{Z_{s_1}^{p_1,q_1}}^\theta, \end{aligned}$$

so by taking the infimum over all such f and g we get

$$\|h\|_{Z_{s_\theta}^{p_\theta,q_\theta}} \le \|h\|_{(Z_{s_0}^{p_0,q_0})^{1-\theta} (Z_{s_1}^{p_1,q_1})^\theta}.$$

Conversely, suppose that $h \in Z^{p_\theta,q_\theta}_{s_\theta}$. Then by Proposition 1.8 there exist $F \in Z^{p_0/(1-\theta),q_0/(1-\theta)}_{s_0(1-\theta)}$ and $G \in Z^{p_1/\theta,q_1/\theta}_{s_1\theta}$ such that $h = FG$ with

$$\begin{aligned}\|h\|_{Z^{p_\theta,q_\theta}_{s_\theta}} &\geq \|F\|_{Z^{p_0/(1-\theta),q_0/(1-\theta)}_{s_0(1-\theta)}} \|G\|_{Z^{p_1/\theta,q_1/\theta}_{s_1\theta}} \\ &= \||F|^{1/(1-\theta)}\|^{1-\theta}_{Z^{p_0,q_0}_{s_0}} \||G|^{1/\theta}\|^{\theta}_{Z^{p_1,q_1}_{s_1}}.\end{aligned}$$

Setting $f = |F|^{1/(1-\theta)}$ and $g = |G|^{1/\theta}$ we see that $f^{1-\theta}g^\theta = |h|$, $f \in Z^{p_0,q_0}_{s_0}$, and $g \in Z^{p_1,q_1}_{s_1}$, so we find that

$$\|h\|_{Z^{p_\theta,q_\theta}_{s_\theta}} \geq \|h\|_{(Z^{p_0,q_0}_{s_0})^{1-\theta}(Z^{p_1,q_1}_{s_1})^\theta}.$$

This completes the proof of (5), and hence also that of Theorem 1.2. □

Remark 1.9. A similar (but much deeper) result for tent spaces was proved by Cohn and Verbitsky [4] (see also [5] for weighted tent spaces incorporating Whitney averages). The spaces $Z^{p,q}_s$ with $q \geq 1$ arise as real interpolants of weighted tent spaces $T^{p,q}_s$ [1, Theorem 2.9], so it is tempting to believe that one could deduce the above results for $q \geq 1$ abstractly. This would be a very inefficient argument, as the Z-space results are much easier to prove than those for tent spaces. Nonetheless this raises the question of whether one could deduce multiplication/factorisation of real interpolants of spaces which themselves have some multiplication/factorisation properties.[1] We will leave this to the experts.

2. Consequences for the Amenta–Auscher monograph

There are a few results in the monograph [2] whose statements are restricted due to the lack of Theorem 1.2. Here we provide a list of these results, and the changes which may be made to their statements in order to incorporate this additional information. Their proofs do not need any modification. We use the notation of [2] without explanation.

- **Proposition 2.31**: the restriction $i(\mathbf{p}), i(\mathbf{q}) \geq 1$ may be removed.
- **Theorems 4.27, 4.31, and 6.42**: the restriction $i(\mathbf{p}_0), i(\mathbf{p}_1) \geq 1$ may be removed in part (ii) of each theorem.
- **Theorem 5.17**: the condition $i(\mathbf{p}) > 1$ may be removed from the second displayed set.
- **Theorem 7.8**: the assumption that $i(\mathbf{p}) > 1$ when $\mathbf{X} = \mathbf{B}$ may be removed. See Remark 7.9.
- **Corollary 7.10**: the assumption that $i(\mathbf{p}), i(\mathbf{q}) > 1$ if $\mathbf{X} = \mathbf{B}$ may be removed.

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[1]Or, going in the other direction, whether one could for example deduce tent space multiplication/factorisation from the corresponding Z-space results. This seems unlikely.

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